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Selberg zeta values of Schottky groups and the Mumford isomorphisms

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§1. Introduction

1.1. Our starting point is the following observation on special values of the Ramanujan delta function

$$\Delta(\tau)=e^{2\pi\sqrt{-1}\tau}\prod_{m=1}^{\infty}\left(1-e^{2\pi\sqrt{-1}m\tau}\right)^{24}$$

on the Poincaré upper half plane $H=\{\tau\in\mathbb{C}\mid \mathrm{Im}(\tau)>0\}$. Let (E,ω) be an elliptic curve with nonzero regular 1-form over a subfield K of $\mathbb{C}$ such that $(E,\omega)\otimes_K\mathbb{C}$ is isomorphic to $\left(\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau_E),2\pi\sqrt{-1}\alpha dz\right)$, where $\tau_E\in H$ and $\alpha\in\mathbb{C}^{\times}$. Then $\Delta(\tau_E)/\alpha^{12}$ is a nonzero element of K. This fact seems due to Shimura, and follows from that $\Delta(\tau_E)/\alpha^{12}$ is the evaluation of the integral modular form $\Delta(\tau)$ of weight 12 on the object (E,ω) over K.

1.2. We consider this fact from another viewpoint using the Mumford isomorphism and a Selberg zeta value of a Schottky group. First, as will be seen in Section 2, $\Delta(\tau)$ gives the Mumford isomorphism for curves of genus 1. Therefore, for the above (E,ω), $\Delta(\tau_E)/\alpha^{12}$ is the evaluation of this isomorphism on (E,ω), and hence belongs to $K^{\times}$. Second, $\Delta(\tau)$ is represented by the Selberg zeta value at 1 of a Schottky group of rank 1 as follows.

A Schottky group is a discrete subgroup Γ of $PGL_2(\mathbb{C})$ which is free of finite rank such that any element of $\Gamma-\{1\}$ is hyperbolic. Then the Selberg zeta value $Z_\Gamma(n)$ of Γ at a positive integer n is defined to be the infinite product

$$Z_\Gamma(n)=\prod_{\{\gamma\}}\prod_{m=n}^{\infty}\left(1-q_\gamma^{m}\right)$$

if this converges, where $\{\gamma\}$ runs through all primitive conjugacy classes in Γ, and q_γ denotes the ratio of eigenvalues of γ such that $|q_\gamma| < 1$. If Γ is a Schottky group of rank 1 which is generated by $\gamma = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$ with $|q| < 1$, then $\Delta(\tau) = (q \cdot Z_\Gamma(1)^{12})\,|_{q=e^{2\pi\sqrt{-1}\tau}}$. Therefore, we have the rationality of

$$\frac{(q \cdot Z_\Gamma(1)^{12})\,|_{q=e^{2\pi\sqrt{-1}\tau}}}{\alpha^{12}}$$

for the above (E, ω). The aim of this note is to give its higher rank version as an analog to a conjecture of Deligne [D] on the rationality of critical values of L-functions of motives divided by periods. More precisely, we use the Mumford isomorphism to study the rationality of the invariant

$$\frac{Z_\Gamma(n)}{\text{period of } n\text{-forms}}$$

for the algebraic curve uniformized by a Schottky group Γ. Actually, we show the rationality of certain ratios of such invariants.

Remark. Strictly speaking, $Z_\Gamma(n)$ is not the value at n of the Selberg zeta function of Γ which becomes

$$\prod_{\{\gamma\}} \prod_{m=n}^{\infty} (1 - |q_\gamma|^m) .$$

However, if Γ is contained in the image of the natural homomorphism $SL_2(\mathbb{R}) \to PGL_2(\mathbb{C})$, then $Z_\Gamma(n) = \prod_{\{\gamma\}} \prod_{m=n}^{\infty} (1 - |q_\gamma|^m)$.

§2. Mumford isomorphism

2.1. Let g and n be positive integers. Denote by $\mathcal{M}_g$ the moduli stack over $\mathbb{Z}$ of (proper and smooth) curves of genus g, and by $\pi : \mathcal{C}_g \to \mathcal{M}_g$ the universal curve of genus g. Then the nth tautological line bundle on $\mathcal{M}_g$ is defined as

$$\lambda_n = \det\left(\pi_*\left(\Omega^n_{\mathcal{C}_g/\mathcal{M}_g}\right)\right)$$

whose fiber at each curve is the determinant of the space of the regular n-forms. Put $d_n = 6n^2 - 6n + 1$. By a basic result of Mumford [Mu], there exists an isomorphism $\mu_{g,n} : \lambda_n \xrightarrow{\sim} \lambda_1^{\otimes d_n}$, namely a nowhere vanishing section of $\lambda_n^{\otimes(-1)} \otimes \lambda_1^{\otimes d_n}$ which is uniquely determined up to a sign under a certain boundary condition when $g \leq 2$. We call $\mu_{g,n}$ the Mumford isomorphism and the Mumford form respectively.

2.2. If $g \leq 4$, then there are explicit formulas for $\mu_{g,2}$ as follows:

- If $g = 1$, then λ_2 is isomorphic to λ_1, and via this isomorphism, $\mu_{1,2}$ is obtained by multiplying $\Delta(\tau)$.
- If $g = 2$, then by taking product of two sections, we have an isomorphism
$$\mathrm{Sym}^2\left(\pi_*\left(\Omega_{\mathcal{C}_2/\mathcal{M}_2}\right)\right) \xrightarrow{\sim} \pi_*\left(\Omega^2_{\mathcal{C}_2/\mathcal{M}_2}\right)$$
which gives rise to an isomorphism $\lambda_1^{\otimes 3} \xrightarrow{\sim} \lambda_2$. This together with $\mu_{2,2}$ becomes an isomorphism $\lambda_1^{\otimes 3} \xrightarrow{\sim} \lambda_1^{\otimes 13}$ obtained by multiplying $(\theta_2/2^6)^2$. Here θ_g denotes the product of even theta constants of degree g.
- If $g = 3$, then as in the genus 2 case, we have a natural homomorphism $\lambda_1^{\otimes 4} \to \lambda_2$ which is an isomorphism outside the hyperelliptic locus. This together with $\mu_{3,2}$ becomes a homomorphism $\lambda_1^{\otimes 4} \to \lambda_1^{\otimes 13}$ obtained by multiplying $\sqrt{-\theta_3/2^{28}}$ (cf. [I1]).
- If $g = 4$, then $\mu_{4,2}$ is represented as $\partial/\partial q_{ij}$ of the Schottky modular form with integral and primitive lowest term (cf. [I2]).

§3. Schottky uniformization

3.1. By Schottky uniformization theory, for a Schottky group Γ of rank g, its region $\Omega_\Gamma \subset \mathbb{P}^1_{\mathbb{C}}$ of discontinuity modulo the action of Γ becomes a Riemann surface of genus g which we denote by X_Γ. Let $\Gamma = \langle \gamma_1, \ldots, \gamma_g \rangle$ be a marked Schottky group, i.e., a Schottky group with free generators. Then there are mutually disjoint closed domains $D_{\pm 1}, \ldots, D_{\pm g} \subset \mathbb{P}^1_{\mathbb{C}}$ bounded by Jordan curves $\partial D_{\pm i}$ such that $\gamma_i(\mathbb{P}^1_{\mathbb{C}} - D_{-i})$ is the interior of D_i for any $1 \leq i \leq g$. Then X_Γ is obtained from $\mathbb{P}^1_{\mathbb{C}} - \bigcup_{i=1}^g D_{\pm i}$ by identifying ∂D_i and ∂D_{-i} via the action of γ_i.

3.2. For a positive integer n, let $\mathbb{C}_{2n-2}[z]$ be the space of polynomials over $\mathbb{C}$ of z with degree $\leq 2n-2$. Then $PGL_2(\mathbb{C})$ acts on $\mathbb{C}_{2n-2}[z]$ as
$$\gamma(\eta(z)) = \eta(\gamma(z)) \cdot \gamma'(z)^{1-n} \quad (\gamma \in PGL_2(\mathbb{C}),\ \eta(z) \in \mathbb{C}_{2n-2}[z]),$$

and we have the Eichler cohomology group $H^1(\Gamma, \mathbb{C}_{2n-2}[z])$. This space has dimension g if $n = 1$, and $(2n-1)(g-1)$ if $n > 1$. Furthermore,

- $H^1(\Gamma, \mathbb{C})$ has a base $\{\eta_1, ..., \eta_g\}$ satisfying that $\eta_i(\gamma_j)$ is Kronecker's delta δ_{ij}.
- $H^1(\Gamma, \mathbb{C}_{2n-2}[z])$ $(n > 1)$ has a base $\{\eta_1, ..., \eta_{(2n-1)(g-1)}\}$ given by

$$\{\eta_{1,n-1}, \eta_{2,1}, ..., \eta_{2,2n-2}, \eta_{k,0}, ..., \eta_{k,2n-2} \ (3 \le k \le g)\}$$

  satisfying that $\eta_{k,j}(\gamma_l)$ is $\delta_{2l}(z-1)^j$ if $k = 2$, and $\delta_{kl}z^j$ if $k \neq 2$.

3.3. We review results of Mcintyre and Takhtajan [MT]. They introduce the period pairing

$$\Psi : H^0\left(X_\Gamma, \Omega_{X_\Gamma}^n\right) \times H^1(\Gamma, \mathbb{C}_{2n-2}[z]) \to \mathbb{C}$$

given by

$$\Psi(\varphi(dz)^n, \eta) = \frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^{g} \oint_{\partial D_i} \varphi \cdot \eta(\gamma_i) dz,$$

where z denotes the natural coordinate on $\Omega_\Gamma \subset \mathbb{P}^1_{\mathbb{C}}$. Then it is shown in [MT] that Ψ is well-defined and nondegenerate. Therefore, there exists a base $\left\{\varphi_i^{(n)}\right\}$ of $H^0\left(X_\Gamma, \Omega_{X_\Gamma}^n\right)$ such that

$$\Psi\left(\varphi_i^{(n)}(dz)^n, \eta_j\right) = \delta_{ij}.$$

3.4. For a marked Schottky group $\Gamma = \langle \gamma_1, ..., \gamma_g \rangle$, we define the modified Selberg zeta value $MZ_\Gamma(n)$ as follows. If $n = 1$, then the infinite product $Z_\Gamma(1)$ is absolutely convergent when $|q_{\gamma_1}|, ..., |q_{\gamma_g}|$ are sufficiently small, and hence it gives a holomorphic function on a nonempty open subset of the Schottky space classifying marked Schottky groups of rank g. It is shown in [MT] that this function can be continued to the whole Schottky space as a holomorphic function, and we define $MZ_\Gamma(1)$ as its value at $\Gamma = \langle \gamma_1, ..., \gamma_g \rangle$. If $n > 1$, then $Z_\Gamma(n)$ always converges, and we define

$$MZ_\Gamma(n) = Z_\Gamma(n)(1 - q_{\gamma_1})^2 \cdots (1 - q_{\gamma_1}^{n-1})^2 (1 - q_{\gamma_2}^{n-1})^2$$

such as $\pm MZ_\Gamma(n) \cdot \bigwedge_i \varphi_i^{(n)}$ is invariant under permutations of $\gamma_1, ..., \gamma_g$.

§4. Mumford forms as infinite products

4.1. We have the following expression of the Mumford form.

Theorem 1. *Assume $g, n > 1$.*

(1) *Under the local trivializations of λ_1 and λ_n by $\varphi_1^{(1)} \wedge \cdots \wedge \varphi_g^{(1)}$ and $\varphi_1^{(n)} \wedge \cdots \wedge \varphi_{(2n-1)(g-1)}^{(n)}$ respectively, there exists a nonzero constant $c_{g,n}$ such that*

$$c_{g,n} \cdot \mu_{g,n} = MZ_\Gamma(1)^{d_n} / MZ_\Gamma(n).$$

(2) *$c_{g,n} \in \mathbb{Z}$ and $c_{g,2} = \pm 1$.*

4.2. We give a sketch of this proof. The proof of (1) uses formulas of Zograf [Z] and of Mcintyre-Takhtajan [MT]. Let S denote the classical Liouville action, and $|| * ||_{\mathrm{Q}}$ denote the Quillen metric. Then by results of [Z] and [MT],

$$\exp(S/12\pi) = \text{const.} |MZ_\Gamma(1)|^2 \cdot ||\varphi_1 \wedge \cdots \wedge \varphi_g||_{\mathrm{Q}}^2,$$

and

$$\exp(S/12\pi)^{d_n} = \text{const.} |MZ_\Gamma(n)|^2 \cdot ||\varphi_1 \wedge \cdots \wedge \varphi_{(2n-1)(g-1)}||_{\mathrm{Q}}^2.$$

Then $\mu_{g,n} = MZ_\Gamma(1)^{d_n} / MZ_\Gamma(n)$ up to constant which implies (1).

The proof of (2) uses the theory of generalized Tate curves given in [I1]. A generalized Tate curve $\mathcal{C}$ gives the universal deformation of degenerate curves with fixed dual graph. More precisely, $\mathcal{C}$ is a stable curve over the ring A of formal power series of deformation parameters whose coefficient ring is the affine coordinate ring over $\mathbb{Z}$ of the moduli space of these degenerate curves. Since $\mathcal{C}$ is Mumford uniformized by a Schottky group over A, by specializing the moduli and deformation parameters, $\mathcal{C}$ becomes Schottky uniformized Riemann surfaces and Mumford curves over local fields. Then $\left\{\varphi_1^{(1)}, \ldots, \varphi_g^{(1)}\right\}$ gives a base of regular 1-forms on $\mathcal{C}$, and $\Psi\left(\varphi_{i_1}^{(1)} \cdots \varphi_{i_n}^{(1)}, \eta_j\right)$ has universal expression as an element of A. For a field k of any characteristic, put $\mathcal{C}_k = \mathcal{C} \otimes_A \mathrm{Frac}(A \otimes k)$, where $\mathrm{Frac}(A \otimes k)$ denotes the field of fractions of $A \otimes k$. Then a theorem of Max Noether implies that $\left\{\varphi_{i_1}^{(1)} \cdots \varphi_{i_n}^{(1)} \,\middle|\, 1 \leq i_k \leq g\right\}$ contains a base of regular n-forms on $\mathcal{C}_k$ if $g \geq 3$, and hence $c_{g,n} \in \mathbb{Z}$. Similar statements hold when $g = 2$

using Noether's theorem for singular curves (cf. [M]). Further, one can show that $\left\{\varphi_{i_1}^{(1)}\varphi_{i_2}^{(1)}\right\}$ contains a base of regular 2-forms on $\mathcal{C}_k$ which is dual to $\{\eta_j\}$ with respect to Ψ, and hence $c_{g,2} = \pm 1$.

§5. Rationality of Selberg zeta values

5.1. We consider certain invariants connected with Selberg zeta values for Schottky uniformized algebraic curves. For a marked Schottky group $\Gamma = \langle \gamma_1, ..., \gamma_g \rangle$ of rank $g > 1$, and a base $\left\{\omega_i^{(n)}\right\}$ of $H^0\left(X_\Gamma, \Omega_{X_\Gamma}^n\right)$, let

$$I(X_\Gamma; n) = \frac{MZ_\Gamma(n)}{\det\left(\Psi\left(\omega_i^{(n)}, \eta_j\right)\right)_{i,j}} \neq 0$$

denote the associated invariant.

Theorem 2. *Assume $n > 1$ and the following conditions:*

- *X_Γ has a model $\mathcal{X}_\Gamma$ as a proper smooth curve over a subring R of $\mathbb{C}$.*
- *$\left\{\omega_i^{(m)}\right\}$ is an R-base of $H^0\left(\mathcal{X}_\Gamma, \Omega_{\mathcal{X}_\Gamma}^m\right)$ for $m = 1, n$.*

Then

$$\frac{I(X_\Gamma; 1)^{d_n}}{I(X_\Gamma; n)} \in R - \{0\} \quad \textit{and} \quad \frac{I(X_\Gamma; 1)^{13}}{I(X_\Gamma; 2)} \in R^\times.$$

Remark. The latter ratio is called the discriminant of $\mathcal{X}_\Gamma$ and studied by T. Saito [S].

5.2. We give a sketch of this proof. Since $I(X_\Gamma; 1)^{d_n}/I(X_\Gamma; n)$ is equal to $c_{g,n}$ times the evaluation of $\mu_{g,n}$ on $\left(\mathcal{X}_\Gamma, \left\{\omega_i^{(m)}\right\}\right)$ for $m = 1$ and n, Theorem 1 implies that this ratio belongs to $R - \{0\}$ and to $R^\times$ if $n = 2$.

5.3. Theorem 2 has a weak point that it seems difficult to give concrete examples of Schottky groups uniformizing algebraic curves defined over number fields. At least, the author does not know examples of number fields K and Schottky groups Γ in $PGL_2(K)$ such that X_Γ is defined over number fields. Then we consider its p-adic version, and we show that a similar assertion as in Theorem 2 holds for a *nonhyperelliptic* Shimura curve which is Cerednik-Drinfeld uniformized by a p-adic Schottky group (cf. [C, Dr]).

Let B be a definite quaternion algebra over $\mathbb{Q}$ which splits at a prime p, and Γ be a congruence subgroup in a maximal $\mathbb{Z}[1/p]$-order of B. Denote by

$$C_\Gamma = \left(\widehat{\overline{\mathbb{Q}}}_p - \mathbb{Q}_p\right)/\Gamma$$

the Shimura curve over a number field K which is Cerednik-Drinfeld uniformized by Γ. We define $MZ_{\Gamma,p}(n)$ as the p-adic modified Selberg zeta value of $\Gamma \subset PGL_2(\mathbb{Q}_p)$, Ψ_p as the p-adic period pairing given by Schneider [Sc] and de Shalit [dS].

Theorem 3. *Let the notation be as above, and assume C_Γ is not hyperelliptic and proper smooth over a subring R of K. We define the p-adic invariant $I_p(C_\Gamma, n)$ as*

$$\frac{MZ_{\Gamma,p}(n)}{\det\left(\Psi_p\left(\omega_i^{(n)}, \eta_j\right)\right)_{i,j}}$$

for an R-base $\left\{\omega_i^{(n)}\right\}$ of $H^0\left(C_\Gamma, \Omega_{C_\Gamma}^n\right)$. Then for $n > 1$,

$$\frac{I_p(C_\Gamma;1)^{d_n}}{I_p(C_\Gamma;n)} \in R - \{0\} \quad \textit{and} \quad \frac{I_p(C_\Gamma;1)^{13}}{I_p(C_\Gamma;2)} \in R^\times.$$

5.4. This proof is similar to the case of Theorem 2 since the assumption and Noether's theorem imply that $H^0\left(C_\Gamma, \Omega_{C_\Gamma}^n\right)$ has a base generated by regular 1-forms which have universal expression.

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